On the Connectivity Assumption of Non-Linear Flocking Models

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***Abstract*— The problem of collective behavior of autonomous agents is discussed. Based on state dependent communication graphs we analyze a family of non-linear flocking models by establishing sufficient initial conditions so as connectivity and thus asymptotic flocking is ensured. We discuss models with non-linear uniformly bounded connection rates with and without delays.**

I. Introduction

Collective self-driven motion of autonomous agents such as flocking of birds , schooling of fishes, swarming of bacteria, appears in many contexts and has been the center of attention of various scientific communities for very long time. The study of flocking mechanisms based on mathematical models was initiated with the seminal work of [15] and was given a rigorous interpretation in [8], paving the way for a broad research field on distributed consensus and flocking algorithms, which is of great interest to both the control and the applied mathematics communities (see for example [14], [5], [7], [13] and references therein).

The control community approach on collective behaviour emanating from distributed calculations out of local interactions between autonomous agents, introduces mathematical models known as consensus algorithms where persistent convex averaging of the agents proposed quantities among them, converges asymptotically to a common value.

In the flocking approach, each agent is characterized by its position and velocity vectors, second order consensus algorithms (speed averaging) yield under, certain assumptions, asymptotic flocking, i.e. co-ordination of the speed of every agent and bounded distance between each other so that the flock is overall bounded when the number of agents is finite.

A. Related work and contribution

Motivated by non-linear flocking models [5], [7] and recent results in opinion dynamics [4], in this work we analyse the asymptotic behaviour of a non-smooth, non-linear flocking model with bounded interactions range. Similar intermediate results have very recently appeared in the literature. In [9] the authors study first order non-linear consensus systems using integral equations and establish similar asymptotic results. In this work, we discuss a second order problem, with different approach (differential inclusions) and aim to answer different questions.

Our major contribution is that we eliminate the assumption of connectivity and we establish initial condition requirements for it, so that asymptotic flocking can occur. Our results extend to the case where there is delay in the communications between agents.

The paper is organized as follows: In section 2, notations, preliminary definitions, background theory and the general model is introduced. In section 3 we establish asymptotic results in the case of no delay in communication and we focus on the initial conditions which guarantee asymptotic flocking with only one flock. In section 4 we derive similar results in case of a uniform constant delay. The discussion of the results is discussed in section 6.

II. Theoretical Framework

In this section, we introduce the background theory used throughout this work.

A. Notations

Consider $N < \infty$ autonomous agents in a d-dimensional Euclidean space. Each agent is characterized by its position and velocity vectors $x_i, v_i \in \mathbb{R}^d$. We write $\mathbf{x} = (x_1, \dots, x_N)'$, $\mathbf{v} = (v_1, \dots, v_N)'$. Also denote $[N] := [1, \dots, N]$. Define the diagonal subspace of $(\mathbb{R}^d)^N$

$$\Delta := \{\mathbf{v} \in (\mathbb{R}^d)^N | v_1 = v_2 = \cdots = v_N\}$$

and its orthogonal complement $\Delta^\perp$. Then any element $\mathbf{v}$ can be uniquely written as $\mathbf{v} = \mathbf{v}_\Delta + \mathbf{v}_{\Delta^\perp}$. This remark will be of essence in the analysis to follow.

Our space is equipped with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^N x_i^T y_i$ where $x_i^T y_i = \sum_{j=1}^d x_i^{(j)} y_i^{(j)}$. Also the norms to be used are the d-dimensional

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$|x_i| := \sqrt{\sum_{j=1}^{d}(x_i^{(j)})^2}$, generalized in the $(\mathbf{R}^d)^N$ space $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} = \sqrt{\sum_{i=1}^{N}|x_i|^2}$ and also the infinity norm $||\mathbf{x}||_\infty = \max_i\{|x_i|\}$. For any two non-empty subsets S_1, S_2 of $[N]$ define the distance $d(S_1, S_2) := \min_{i\in S_1, j\in S_2}|x_i - x_j|$. Finally any time derivative is taken to be the upper right Dini derivative.

B. Algebraic Graph Theory

A weighted graph is a triple $G = (V, E, A)$ where V is a finite set of nodes and from now on $V = [N]$, E is a subset of $[N] \times [N]$ and A a matrix taking values in $\mathbb{R}_+$ with $a_{ij} > 0$ if and only if $(i, j) \in E$. If the non-zero elements of A are set identically to 1, then we say we have a topological graph and denote it by $\bar{G}$ together with all its related quantities. A graph is said to be connected if for any $i, j \in [N]$ there exists a subset of E of non-disjoint elements of $[N]$, with a pair containing i and a pair containing j. The matrix representation of any graph G is achieved with the adjacency matrix A, the diagonal matrix $D := \mathrm{Diag}[\sum_j a_{ij}]$ and the combinatorial Laplacian, $L := D - A$ (see [2]).

1) Spectral properties of A and L: In this work A is symmetric with N real eigenvalues the maximum of which is the spectral radius of the graph. The connectivity of G is characterized by the *Fiedler number*, ϕ, of A which is the second smallest eigenvalue of L. It is a well known result that G is connected if and only if $\phi > 0$ whereas in general the algebraic multiplicity of the zero eigenvalue of L equals the number of connected components of G [2]. Another important fact from the algebraic graph theory is that $\ker(L) = \Delta$. The next proposition establishes a useful connection between geometric and topological graphs.

Proposition 1 ([2], [10]): Given a connected graph $\bar{G}$ with diameter $\bar{\rho}$ let $\bar{\lambda}_A$ and $\bar{\phi}$ be the largest eigenvalue of A and the second smallest eigenvalue of L. Then

$$\bar{\lambda}_A \le \sqrt{\frac{2|E|(N-1)}{N}} \le N - 1 \tag{1a}$$

$$\bar{\phi} \ge \frac{4}{N\bar{\rho}} \ge \frac{4}{N(N-1)} \tag{1b}$$

C. Model Description

The dynamical system in consideration is

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \sum_{j=1}^{N} a(|x_i - x_j|)(\hat{v}_j - v_i) \end{aligned} \quad , \quad i = 1 \dots N \tag{2}$$

where $\hat{v}_j = v_j(t - c)$, and c is a fixed non-negative number modeling a uniform time-invariant delay. Initially in section 3 we set $c = 0$ for all i and a non zero delay will be considered in section 4.

Each agent is equipped with a communication rate function with which it can exchange information with any other agent located within distance R. That is $a(r) : \mathbb{R}_+ \to [0, M]$ is a non-negative, non-increasing scalar valued function which models the communication rate between agents in the 2^{nd} order consensus algorithm (2). The function a models the effect agents have with each other so that the further away two agents lie from each other the weaker will be the effect between one another. Furthermore, after a fixed distance $R < \infty$ we assume no communication. In this work we will consider uniform bounds of connectivity: a is taken to be Lipschitz continuous with bounded support i.e. $a(z) = 0$ if and only if $z \ge R$ and $\min_{0\le z\le R} a(z) =: a_R > 0$. This framework is in agreement with the classical connectivity assumption in distributed algorithms: If two agents are connected the connection weight must be uniformly lower bounded.

This setting does not yield smooth dynamics, hence generalized notions of solutions to (2) need to be used. Before we introduce these solutions we remark that the connectivity graph depends on the rate functions and consequently on the relative positions of agents in the sense that i and j exchange information if and only if $|x_i - x_j| < R$.

A commonly accepted definition of flocking with one connected component is proposed in [7] and will be considered in this work:

Definition 1: Consider N agents following the dynamics of (3). We say that the system exhibits asymptotic flocking if

$$\lim_t |v_i(t) - v_j(t)| = 0 \quad , \quad \sup_{0\le t<\infty} |x_i(t) - x_j(t)| < \infty$$

for all $i, j \in [N]$.

We derive initial conditions so that the flock will remain connected for all times as it coordinates its velocity, in agreement with the definition above.

D. Krasovskii's solutions to ODEs

Consider the initial value problem

$$\dot{y} = f(y) \quad , \quad y(t_0) = y_0 \tag{KRS}$$

where $y : \mathbb{R} \to \mathbb{R}^l$, $f : \mathbb{R}^l \to \mathbb{R}^l$ and $y_0 \in \mathbb{R}^l$. A Krasovkii solution to (KRS) on an interval $I \subset \mathbb{R}$ containing t_0 is a map $s : I \to \mathbb{R}^N$ such that: s is absolutely continuous (a.c.) on I, $s(t_0) = y_0$, $\dot{s}(t) \in \mathcal{K}f(s(t))$ where

$$\mathcal{K}f(y) := \bigcap_{\delta>0} \bar{co}\{f(u) : ||u - y|| < \delta\} \tag{OP}$$

is the Krasovksii operator.

In view of the uniform bounds of a, the set E of the connections in G plays a crucial role as it includes the surfaces of discontinuity of (2). We write

$$E(\mathbf{x}) = \{(i, j) \in [N] \times [N] : |x_i - x_j| < R\}$$
$$\partial E(\mathbf{x}) = \{(i, j) \in [N] \times [N] : |x_i - x_j| = R\}$$

Since G is a state dependent graph, at least in the sense of E, we may consider all the related quantities as functions of $\mathbf{x}$. The *augmented graph* $\mathcal{G} := ([N], E(\mathbf{x}) \cup \partial E(\mathbf{x}), W)$ defines the family of graphs $\mathcal{G}_H = ([N], E(\mathbf{x}) \cup H, W)$ for any $H \subset \partial E(\mathbf{x})$.

E. Functional spaces

The objectives of this work induce the following definitions. For $-c \le t \le 0$ we define the Banach space $\mathcal{C}_{[-c,0]}$ of continuous real functions in $\mathbb{R}^{Nd}$ with the norm, $||\mathbf{v}||_{\mathcal{C}} := \sup_{t\in[-c,0]} ||\mathbf{v}(t)||$. Then let $\mathcal{F}$ be a set valued map defined on $[0,\infty) \times \mathbf{R}^{2Nd} \times C_{[-c,0]}$ that takes convex values. The associated Cauchy problem is, for the purposes of this paper, stated as follows

$$\begin{aligned}
&\dot{\mathbf{x}} = \mathbf{v} \\
&\dot{\mathbf{v}} \in \mathcal{F}(\mathbf{x}, \mathbf{v}, \mathbf{v}_t) \qquad t \ge 0 \\
&\mathbf{x}(-c) \in \mathbf{R}^{Nd} \quad , \quad \mathbf{v}(t) \in \mathcal{C}_{[-c,0]} \quad \forall \ t \in [-c, 0]
\end{aligned}$$

where $\mathbf{v}_t = \hat{\mathbf{v}} = \mathbf{v}(t-c)$. The existence of solutions in these types of problems is addressed in [3] where the assumptions on $\mathcal{F}$ coincide with the Krasovskii operator (OP).

III. Flocking without delays

In this section, asymptotic results will be established for (2) without delays. The case of delays will be studied in section 4.

A. Preliminary results

Using the vectors $\mathbf{x}$ and $\mathbf{v}$ we rewrite (2) as

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v} \\
\dot{\mathbf{v}} &= -L_{\mathbf{x}}\mathbf{v}
\end{aligned} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0, \mathbf{v}(0) = \mathbf{v}_0 \tag{3}$$

where $L_{\mathbf{x}}$ is the *expanded Laplacian* which is a mapping that acts on $(\mathbb{R}^d)^N$ such that $L_{\mathbf{x}} = L \otimes I_d$. Consequently, all the properties of L are carried over to $L_{\mathbf{x}}$. The next proposition verifies the well-posedness of solutions together with elementary properties.

Remark 1: In view of the discussion in section 2.1 flocking between any two agents is equivalent to $v_i - v_j \to 0$ and $|x_i - x_j| < \infty$. Since the kernel of $L_{\mathbf{x}}$ is Δ the analysis will be focused on $\mathbf{x}_{\Delta^\perp}, \mathbf{v}_{\Delta^\perp}$ which due to the symmetry of a, obey the same dynamics as (3).

Proposition 2: The following statements are true for any Krasovskii solution $(\mathbf{x}(t), \mathbf{v}(t))$ of (3)

1) All solutions exist and are defined for all times
2) The solutions satisfy the differential inclusion
$$\dot{\mathbf{v}}(t) = -\sum_{H\subset\partial E(\mathbf{x}(t))} \alpha_H^{\mathbf{x}}(t) L_H(\mathbf{x}(t))\mathbf{v}(t)$$
for any $\alpha_H(t) \ge 0$, $\sum_H \alpha_H(t) \equiv 1$
3) $\mathbf{v}_{av}(t) := \frac{1}{N}\sum_1^N v_i(t) \equiv \mathbf{v}_{av}(0)$.

Proof: (1) follows trivially from the fact that solutions are bounded in $\mathbf{v}$ and have a sub-linear growth in $\mathbf{x}$. The result follows by standard arguments in [6]. (2) follows from (OP), the observation of the fact that the discontinuity hyper-surfaces are of zero measure, the finite number of agents and Caratheodory's theorem on convex hulls (see for example [12]). Finally (3) is calculated as follows:

$$\begin{aligned}
\dot{\mathbf{v}}_{ave} &= \frac{1}{N}\sum_{i=1}^N \dot{v}_i \\
&= -\frac{1}{N}\sum_i\sum_H \alpha_H^{\mathbf{x}}(t) \sum_{i\sim j} a(|x_i - x_j|)(v_i - v_j) \\
&= -\frac{1}{N}\sum_H \alpha_H^{\mathbf{x}}(t) \sum_i\sum_{i\sim j} a(|x_i - x_j|)(v_i - v_j) \\
&= 0
\end{aligned}$$

due to the symmetry of the weights and the fact that $i \sim j \Leftrightarrow j \sim i$. ■

1) Convergence to multiple flocks: The limit set of (3) will consist of connected components of (most likely) different speed from one another that are at least R away. Consider the subset

$$\Omega = \big\{(\mathbf{x}, \mathbf{v}) : v_i \ne v_j \Rightarrow |x_i - x_j| \ge R\big\} \tag{4}$$

Theorem 1: All Krasovskii solutions $(\mathbf{x}(t), \mathbf{v}(t))$ of (3) converge to a point $(\bar{x}, \bar{v}) \in \Omega$.

Proof: The proof consists of the next three steps.

1) Ω is a closed (3)-weakly invariant set. The first part follows trivially from the definition of Ω. For the second part consider any point of Ω. Then one can uniquely construct the communication graph with k connected components in each of which the agents share a common value in their velocity. It follows that for all agents in the same connected component, their velocities are identical for all times and consequently their relative positions. Hence, all initial components stay the same and solutions remain in Ω for all times, in particular in the same element of Ω.
2) Ω is globally asymptotically stable. This will be proved with the use of the quadratic Lyapunov function $V(\mathbf{v}) = \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle$. Then
$$\begin{aligned}
\dot{V} &= -\sum_{H\in\partial E(\mathbf{x})} \alpha_H^{\mathbf{x}}(t)\langle L_H(\mathbf{x}(t))\mathbf{v}(t), \mathbf{v}(t)\rangle \\
&= -\sum_{H\in\partial E(\mathbf{x})} \alpha_H^{\mathbf{x}}(t)\sum_{i\sim j} a(|x_i - x_j|)(v_i - v_j)^2 \\
&\le -\sum_{H\in\partial E(\mathbf{x})} \alpha_H^{\mathbf{x}}(t)\frac{a_R}{2}\sum_{i\sim j\in V_H}(v_i - v_j)^2 \le 0
\end{aligned}$$
The equality holds if and only if $(\mathbf{x}, \mathbf{v}) \in \Omega$. The result follows from La-Salle's invariance principle for differential inclusions (see [1]) and arbitrary initial conditions.
3) Convergence to a single equilibrium point in Ω. Consider any element of the set of the Krasovskii

solutions. These solutions create k, connected components, V_k where $1 \leq k \leq N$. Let $t_1, \ldots t_k$ be the times of creation of each of these components. Then for any agent $i \in V_l(t), t \geq t_l$ $v_i \rightarrow \frac{1}{|V_l|}\sum_{l\in V_k} v_l(t_k)$ by Proposition 2 and the fact that the system is autonomous. Since different flock average speeds yield asymptotically unbounded position differences the result follows.

■

Remark 2: Ω is weakly invariant in the sense that there exist initial conditions in Ω that would allow some Krasovksii solution to lie out of Ω (for more, see [6]).

Remark 3: One can think of Ω as the "ω-limit set" of (3) if it could be restricted to the variable $\mathbf{v}$. It is intuitive that the actual ω limit set is non-empty on condition that $\mathbf{x}$ remains bounded, which can happen only if all N agents are connected or under special initial conditions, e.g. $d = 1, N = 3$ and $|x_1(0) - x_2(0)| > R$ and $v_1(0) = v_2(0) = v_3(0)$.

B. Initial conditions for global flocking

In this section we will focus on estimating the subset of the phase space such that initial conditions from this subset guarantee asymptotic flocking with one connected component. In view of the metric function introduced in section 2.1 we can easily derive the following result (stated without proof).

Lemma 1: The flock of N agents consists of a single connected component if and only if

$$\max_{S\subset[N]} d(S,S^c) < R \tag{5}$$

1) Bounds on connectivity: In this section, we derive bounds on the connectivity of (3).The Fiedler number intriduces in section 2.2 is defined as

$$\phi_x := \min_{\mathbf{v}\neq 0,\mathbf{v}\in\Delta^\perp} \frac{\langle L_\mathbf{x}\mathbf{v},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle}$$

Proposition 3: For $N \geq 2$, it holds that

$$\phi_x \geq \frac{4}{N(N-1)} a_R$$

Proof: Using the symmetry and the monotonicity of $a(|x_i - x_j|)$ we have

$$\begin{aligned}
\frac{\langle L_\mathbf{x}\mathbf{v},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle} &= \frac{1}{2}\frac{\sum_{i,j=1}^N a_{ij}|v_i - v_j|^2}{\langle \mathbf{v},\mathbf{v}\rangle} \\
&\geq \min_{ij}{}^{+} a(|x_i - x_j|)\frac{\frac{1}{2}\sum_{i,j=1}^N |v_i - v_j|^2}{\langle \mathbf{v},\mathbf{v}\rangle} \\
&= a_R \frac{\langle \bar{L}_x\mathbf{v},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle} \\
&\geq a_R \min_{\mathbf{v}\in\Delta^\perp,\mathbf{v}\neq 0} \frac{\langle \bar{L}_x\mathbf{v},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle} = a_R\bar{\phi}
\end{aligned}$$

The result follows from Proposition 1 and the upper bound $\rho \leq N - 1$. ■

2) State equations for uniform a and conditions for global flocking: We need to derive the state equations in terms of the function

$$\begin{aligned}
\eta(\mathbf{x}(t)) :&= \max_{S\subset[N]} d(S,S^c)(t) \\
&= \max_{S\subset[N]} \min_{i\in S, j\in S^c} |x_i(t) - x_j(t)|
\end{aligned} \tag{6}$$

One can assume that throughout a solution of (3) agents may enter or leave the radius range of each other. This creates discontinuities in the dynamics of velocities hence η is continuous and piecewise differentiable. Assuming (5) initially holds, then by continuity there exists $\tau > 0$ such that the communication graph is connected for $t \in [0,\tau)$.

Then for almost all t in this interval

$$\begin{aligned}
\frac{d}{dt}\eta(\mathbf{x}) &= \frac{d}{dt}\min_i \max_j |x_i - x_j| \\
&\leq \min_i \max_j |\frac{d}{dt}(x_i - x_j)| \\
&\leq 2\max_i |v_i| = 2||\mathbf{v}||_\infty \leq 2||\mathbf{v}||
\end{aligned} \tag{7}$$

by the equivalence of Euclidean norms. The step of putting the time derivative inside the $\min$, $\max$ operators is justified after picking the agents that satisfy this extremum. If there are more than one solutions, we pick the solution with the maximum difference of velocities. If there are still more than one solutions, then we pick any of them by chance and stick with it until the time of a possible new solution.

This is a sufficient, although rough, estimate for the evolution of η. The solution of $\mathbf{v}$ satisfies the form of Proposition 2 and for $t \in [0,\tau)$ we can establish:

$$\begin{aligned}
\frac{d}{dt}||\mathbf{v}||^2 &= -2\sum_{H\subset\partial E(\mathbf{x}(t))} \alpha_H^\mathbf{x}(t)\langle L_H(\mathbf{x})\mathbf{v},\mathbf{v}\rangle \\
&\leq -2\sum_{H\subset\partial E(\mathbf{x}(t))} \alpha_H^\mathbf{x}(t)\bar{\phi}(\mathbf{x})a_R||\mathbf{v}||^2 \Rightarrow \\
\frac{d}{dt}||\mathbf{v}|| &\leq -\sum_{H\subset\partial E(\mathbf{x}(t))} \alpha_H^\mathbf{x}(t)\bar{\phi}(\mathbf{x})a_R||\mathbf{v}|| \\
&\leq -\frac{4}{N(N-1)}a_R||\mathbf{v}||
\end{aligned} \tag{8}$$

The system (7), (8) of the differential inequalities is defined in the subset of $\mathbf{R}_+, \mathbf{R}^{Nd} \times \mathbf{R}^{Nd}$

$$W = \{(t,\mathbf{x},\mathbf{v}) : \eta(t) \leq R\}$$

Since any solution can be continued to the boundary of W the goal is to show that there exist initial conditions such that this boundary will be the one with respect to t.

3) Convergence: Following [7] we consider the Lyapunov-like functionals

$$V(||\mathbf{v}||,\mathbf{x},y) = ||\mathbf{v}|| \pm \frac{2}{N(N-1)}a_R[\eta(\mathbf{x}) - y] \tag{9}$$

Lemma 2: Given any solution of (3), $V(t)$ as defined in (9) are absolutely continuous functions in the interval of the existence of solution.

Proof: Note that $\mathbf{v}(t)$ is a.c. with bounded derivative by Proposition 2. Then $\mathbf{v}(t)$ satisfies a Lipschitz condition in $[0,\tau)$ and so does $||\mathbf{v}(t)||$ (triangular inequality) which in turn makes $||\mathbf{v}(t)||$ absolutely continuous. The result follows the fact that the sum/difference of a.c. functions is a.c. (see the discussion at page 107 of [11]). ∎

We calculate

$$\dot{V}(||\mathbf{v}(t)||,\mathbf{x}(t),\eta_0) = \frac{d}{dt}||\mathbf{v}|| \pm \frac{2}{N(N-1)}a_R\frac{d}{dt}\eta(\mathbf{x}) \\ \leq 0$$

In view of Lemma 2, (7),(8) and the fact that $\dot{V}$ is simply the set-valued derivative of V with respect to the differential inclusion, it follows that $V(t)-V(0) \leq 0$ for every $t \in [0,\tau)$ or

$$||\mathbf{v}(t)|| - ||\mathbf{v}(0)|| \leq -\frac{2a_R}{N(N-1)}\left|\eta(\mathbf{x}(t)) - \eta_0\right| \quad (10)$$

Theorem 2: Assume that initial conditions satisfy

$$||\mathbf{v}(0)|| < \frac{2a_R}{N(N-1)}(R-\eta_0) \quad (11)$$

Then there exists $\tau > 0$ and $y < R$ such that

$$\eta(\mathbf{x}(t)) \leq y < R\ ,\ ||\mathbf{v}(t)|| \leq ||\mathbf{v}(0)||e^{-\frac{4}{N(N-1)}a_R t} \quad (12)$$

for $t \in [0,\tau)$. Moreover $\tau = \infty$ so that asymptotic flocking occurs.

Proof: From (11) the communication network is initially connected, as $\eta_0 < R$. Set $\tau := \inf\{t^* : \eta(\mathbf{x}(t^*)) = R\}$. Then obviously $\eta(\mathbf{x}(t)) < R$ throughout $[0,\tau)$. There exists a unique $z < R$ such that

$$||\mathbf{v}(0)|| = \frac{2a_R}{N(N-1)}(z-\eta_0)$$

Consider the two cases:

- $\eta(\mathbf{x}(t)) < \eta_0 < R$ and the first result follows.
- $\eta(\mathbf{x}(t)) \geq \eta_0$ then for $t \in [0,\tau)$

$$\frac{2a_R}{N(N-1)}(z-\eta_0) = ||\mathbf{v}(0)|| \\ \geq \frac{2a_R}{N(N-1)}(\eta(\mathbf{x}(t)) - \eta_0)$$

so $\eta(\mathbf{x}(t)) \leq z < R$

So by taking $y = \max\{z,\eta_0\}$ we get $\forall t \in [0,\tau)$, $\eta(t) < R$. It follows that τ can become arbitrarily large (see Thm 2, p. 78 [6]). The second assertion of the theorem is a direct result of (8). ∎

a) Example.: Consider $N=2,\ d=1,\ R=1,\ a(z)\equiv 1,\ z\in[0,1)$, for $b\neq 0, -1<g<1$ consider the initial data $x_1(0)=0\ v_1(0)=b$ and $x_2(0)=g,\ v_2(0)=-b$. Then for some $\tau>0$

$$\frac{d}{dt}\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

for $t\in[0,\tau)$. Note that $\eta(t)=|x_1(t)-x_2(t)|$ with $\eta_0=|g|$. After elementary calculations

$$\eta(t) = |x_1(t)-x_2(t)| \leq |g| + \int_0^t |v_1(s)-v_2(s)|ds \\ < |g|+|b| < 1$$

The last step holds precisely when (11) holds.

IV. Flocking with delays

The results of the previous section can be generalized to the case where there are delays in the communications between agents. From (2) one notes that there is delay only in the second order information, i.e. in the velocity of the j^{th} node as this is received from the i^{th} node when i,j are within range R.

The general convergence results will be omitted due to space limitation. We only note what the intuition suggests, i.e. the limit sets coincide with (4). The presence of delay makes the Laplacian approach followed in the previous section obsolete and there is need to establish different bounds. The second part of which (3) is written as follows

$$\dot{\mathbf{v}} = -D_{\mathbf{x}}\mathbf{v} + A_{\mathbf{x}}\hat{\mathbf{v}}$$

Then assuming that (5) is fulfilled in the initial data

$$\frac{d}{dt}||\mathbf{v}|| \leq -a_R||\mathbf{v}|| + M\bar{\lambda_A}||\hat{\mathbf{v}}||$$

where $M = \max_r a(r)$ The last step is justified as follows. Considering the symmetry and the non-negativity of A

$$||A_{\mathbf{x}}\hat{\mathbf{v}}|| \leq ||A_{\mathbf{x}}||||\hat{\mathbf{v}}|| = \sqrt{\lambda_{\max}(A_{\mathbf{x}}^T A_{\mathbf{x}})}||\hat{\mathbf{v}}|| \\ = |\lambda_{\max}(A_{\mathbf{x}})|||\hat{\mathbf{v}}|| \\ \leq M|\bar{\lambda_A}|||\hat{\mathbf{v}}|| \leq M(N-1)||\hat{\mathbf{v}}||$$

using Proposition 1 and basic inequalities from graph theory. The functionals in this case are

$$V(||\mathbf{v}||,\mathbf{x},y) = d_1||\mathbf{v}|| \pm d_2(\eta(\mathbf{x})-y) \\ + d_3\int_{t-c}^t ||\mathbf{v}(s)||ds \quad (13)$$

for some constants $d_i > 0$ to be determined. Throughout any Krasovskii solution

$$\frac{d}{dt}V \leq -d_1 a_R||\mathbf{v}|| + d_1 M\bar{\lambda_A}||\hat{\mathbf{v}}||\pm \\ \pm 2d_2||\mathbf{v}|| + d_3||\mathbf{v}|| - d_3||\hat{\mathbf{v}}|| \leq 0 \quad (14)$$

which in both cases hold if

$$M\bar{\lambda}_A a_R > 1 \tag{15}$$

which is true if $Ma_R > \frac{1}{N-1}$ by Proposition 1, and also if

$$\frac{d_1}{d_3} < M\bar{\lambda}_A \quad , \quad \frac{d_2}{d_3} < \frac{M\bar{\lambda}_A a_R - 1}{2} \tag{16}$$

Eq. (15) is a necessary condition that the system's parameters need to fulfil so that the approach discussed here is valid. Using the same approach as in Theorem 2 we derive

Theorem 3: If the initial conditions satisfy

$$||\mathbf{v}(0)|| < \frac{d_2}{d_1}\left(R - \frac{d_3}{d_2}\int_{-c}^{0} ||\mathbf{v}(s)||ds - \eta_0\right) \tag{17}$$

then (2) guarantees asymptotic flocking.

Proof: The proof follows exactly the one as in Theorem 2 and will be omitted. ■

As it should be intuitively clear the sole validity of (5) is not sufficient in presence of delays. The latter result asserts that global asymptotic flocking is guaranteed under certain initial conditions both of the positions and the velocities of the agents in $[-c, 0]$. The problem is answered with condition (17) which bridges the gap between the admissible initial data and the systems parameters.

It is noted that (17) holds for arbitrary values of bounded c and thus for these values where convergence in a single point of Ω is possible.

V. Concluding remarks

The collective behaviour of self-driven autonomous agents was discussed. Unlike the vast majority of works in the literature, here we did not assume a priori connectivity of any kind. The goal was to derive an estimation of the set of initial data such that the agents would co-ordinate their velocities so as to form a unified flocking body. The analysis includes models without and with delay. The assumptions taken into consideration lie on the symmetry of communication weights so that useful algebraic graph theory results are utilized. Moreover, the weight function was assumed non-increasing in order to model the effect of two agents within distance at most R. However a simply continuous, of bounded support function a would suffice. Another crucial assumption was the uniform bound a_R which causes the discontinuities in the dynamics.

A. Future Work

The future work includes three different directions. The first direction, is towards establishing an efficient approximation algorithm for computing η_0 in case of large N. The cost of the connectivity assumption is a very difficult initial computation, in case the initial setting is arbitrary. We believe that there are broad classes of initial graph topologies which favor efficient computations. The second is studying the same model in the presence of uncertainties, e.g. potential functions which typically model collision avoidance standards, [13]. The third, and most challenging, is the case of non-uniform weights. It should be noted that there is yet to be found a Lyapunov functional to study the convergence of non-linear flocking models (in the sense of Cucker-Smale , [5]) with delays. The Lyapunov-Krasovskii functional defined in (13) is not applicable when the connectivity weights are not bounded from below.

VI. Acknowledgment

This work was partially supported by The Army Research Office under MURI grant W911NF-08-1-0238, the US Air Force Office of Scientific Research MURI grant FA9550-09-1-0538 and by the National Science Foundation (NSF) grant CNS-1035655.

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